Matemáticas para la Economía
Examen Final, 21 de enero de 2014 **SOLUCIÓN**

1

(a) (5 points) Halle el siguiente determinante:

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{vmatrix}$$

(b) (5 points) Sean A, B matrices cuadradas del mismo orden e invertibles. Halle la matriz X que soluciona el sistema matricial siguiente:

$$(A+B)(X+B) = B(X+A+B).$$

Solución:

(a) El determinante es equivalente a

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & -1 & -3 \\ 0 & -3 & 0 & -3 \end{vmatrix} = (-1)\begin{vmatrix} -1 & -3 \\ 0 & -3 \end{vmatrix} - \begin{vmatrix} -2 & -3 \\ -3 & -3 \end{vmatrix} = -3-(6-9) = -3+3 = 0.$$

(b) $X = A^{-1}(BA - AB)$.

2

Sea $a \in \mathbb{R}$. Se considera el sistema lineal

$$\begin{cases} x & -2y & +z & = & 0 \\ -x & +y & +2z & = & 2 \\ x & -3y & +z & = & a \end{cases}$$

(a) (5 points) Discuta el sistema.

(b) (5 points) Resuelva el sistema. ¿Es $(1, 2, 1)$ solución?

Solución:

(a) La matriz A del sistema es cuadrada con determinante nulo. La matriz aumentada A^* es equivalente a

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & a-2 \end{array} \right).$$

Luego A^* tiene el mismo rango que A si y sólo si $a = 2$. Por tanto, el sistema no admite solución si $a \neq 2$. Cuando $a = 2$, el sistema es compatible indeterminado, dado que el rango de A es dos, menor que el número de variables. (Teorema de Rouché).

(b) En el caso $a = 2$, utilizando el sistema escalonado equivalente hallado en el apartado anterior, es fácil ver que las soluciones son de la forma $(5z - 4, 3z - 2, z)$, donde $z \in \mathbb{R}$. Observamos que $(1, 2, 1)$ no es solución, ya que $z = 1$ corresponde con $(1, 1, 1)$, no con $(1, 2, 1)$.

3

Se considera la función $f(x) = \sqrt{\dfrac{4x-1}{x-4}}$.

(a) (5 points) Calcule el dominio y las asíntotas de f.

(b) (5 points) Asumiendo que f admite inversa (no es necesario probarlo), halle la inversa de f y calcule su dominio.

Solución:

(a) The domain is $(4, \infty)$ and $x = 4$ is a vertical asymptote since $\lim_{x \to 4^+} f(x) = \infty$. Since $\lim_{x \to \infty} \frac{\sqrt{4x-1}}{\sqrt{x-4}} = 4$, we have that $y = 4$ is an horizontal asymptote. There is no oblique asymptote.

(b) The inverse exists since it is asserted that the function is strictly monotonous. To find the inverse, we have to solve for x in the equation $\sqrt{\frac{4x-1}{x-4}} = y$. We get $4x - 1 = y^2(x-4)$, that is, $x = \frac{1-4y^2}{y^2-4}$.

4

(a) (3 points) Demuestre que la ecuación $x^2 - 2\sqrt{2x} + x - 1 = 0$ tiene al menos una solución positiva.

(b) (7 points) Se considera la función $f(x) = \dfrac{5x}{1+x^2}$. Estudie los estremos locales y globales de f en el intervalo $[0,2]$.

Solución:

(a) The function $g(x) = x^2 - 2\sqrt{2x} + x - 1$ is continuous in $[0,\infty)$. Note that $g(0) = -1 < 0$ and $g(2) = 1 > 0$, thus by Bolzano's Theorem there is some $0 < c < 2$ such that $g(c) = 0$, that is, there is c which is a positive solution of the proposed equation.

(b) The function is continuous in $[0,2]$, thus according to Weierstras' Theorem, f has global maximum and global minimum in this interval. Moreover, it is derivable, thus we can locate possible extrema by equating the derivative of f to zero.

$$f'(x) = 5\frac{1-x^2}{(1+x^2)^2} = 0 \text{ if and only if } x = \pm 1.$$

Now, $-1 \notin [0,2]$, so it is disregarded. We have three candidates for being extremum of f: 1, 0 and 2. Evaluating we get $f(1) = \frac{5}{2}$, $f(0) = 0$ and $f(2) = 2$. Thus, $x = 0$ is the global minimum and $x = 2$ is the global maximum. Noting that f' is negative at the left of $x = 1$ and positive at the right, we conclude that $x = 1$ is local minimum.

5

Sean $a, b \in \mathbb{R}$. Considere la función $f(x) = \begin{cases} 1 + x^2, & \text{if } x \leq 0; \\ e^{1-2a+bx}, & \text{if } x > 0. \end{cases}$

(a) (4 points) Estudie si f es continua en $x = 0$.

(b) (6 points) Estudie si f es derivable en $x = 0$.

Solución:

(a) Clearly the left sided limit of f at $x = 0$ is 1, and the right sided limit is e^{1-2a}. So, in order to get continuity, $a = \frac{1}{2}$.

(b) Since a differentiable function is also continuous, f is not differentiable at $x = 0$ when $a \neq \frac{1}{2}$. Hence, suppose that $a = \frac{1}{2}$. After plugging this value into the expression of f we see that the left sided derivative is $2x$ and the right sided derivative is be^{bx}. At $x = 0$ these expressions take the value 0 and b, respectively. Thus, f is differentiable at $x = 0$ if and only if $a = \frac{1}{2}$ and $b = 0$.

6

(a) (5 points) Halle la integral de la función $f(x) = x \ln x$. (Pista: use integración por partes.)

(b) (5 points) Halle el área de la región limitada por al gráfica de las funciones $f(x) = x - 1$ y $g(x) = -x^2 + x + 3$.

Solución:

(a) Take $u = \ln x$ and $dv = xdx$. Then $du = \frac{1}{x}$ and $v = \frac{x^2}{2}$. Apply the formula of integration by parts $\int udv = uv - \int vdu$, to get

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$$

(b) $A = \displaystyle\int_{-2}^{2} (g(x) - f(x))dx = \int_{-2}^{2} 4 - x^2 dx = 4x - \frac{x^3}{3}\Big|_{-2}^{2} = \frac{32}{2} u^2.$